



The Open
University

MST124

Essential mathematics 1

Exercise Booklet 11

1 Taylor polynomials of small degree

Exercise 1

- (a) Find the constant Taylor polynomial about 0 for the function $f(x) = e^{-x}$.
- (b) Use this polynomial to write down approximations for $e^{-0.1}$ and $e^{-0.01}$. Use your calculator to find the value of the associated remainder to five decimal places.
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Exercise 2

- (a) Find the constant Taylor polynomial about 1 for the function

$$f(x) = \frac{1}{1+x}.$$

- (b) Use this polynomial to write down approximations for $\frac{1}{2.1}$ and $\frac{1}{1.9}$. In each case, use your calculator to find the value of the associated remainder to five decimal places.
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Exercise 3

- (a) Find the linear Taylor polynomial about 0 for the function $f(x) = e^{-x}$.
- (b) Use this polynomial to find approximations for $e^{-0.1}$ and $e^{-0.01}$. In each case, use your calculator to find the associated remainder to five decimal places.
- (c) How do these remainders compare with those found in Exercise 1(b)?
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Exercise 4

- (a) Find the linear Taylor polynomial about 0 for the function

$$f(x) = (4-x)^{1/2}.$$

- (b) Use this polynomial to find an approximation for $3.98^{1/2}$, and use your calculator to find the value of the associated remainder to eight decimal places.
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Exercise 5

Find the linear Taylor polynomial about 1 for the function $f(x) = e^{-x}$.

Exercise 6

Find the linear Taylor polynomial about 2 for the function

$$f(x) = \frac{1}{1+x}.$$

Exercise 7

Find the quadratic Taylor polynomial about 0 for each of the following functions.

- (a) $f(x) = e^{2x}$ (b) $f(x) = \sin(\frac{1}{2}x)$
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Exercise 8

- (a) Find the quadratic Taylor polynomial about 0 for the function

$$f(x) = x \cos x.$$

- (b) Use this polynomial to find an approximation for $f(0.01)$, and use your calculator to find the value of the associated remainder to eight decimal places.
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Exercise 9

Find the quadratic Taylor polynomial about 1 for each of the following functions.

- (a) $f(x) = e^{2x}$ (b) $f(x) = x^{1/2}$
-

2 Taylor polynomials of any degree

Exercise 10

- (a) Find the quartic Taylor polynomial about 0 for each of the following functions.
- $f(x) = \cos(2x)$
 - $f(x) = \ln\left(\frac{1}{1+x}\right)$
- (b) (i) Observe the pattern of successive derivatives found in part (a)(ii) and use this to write down a formula for the n th derivative of $f(x) = \ln(1/(1+x))$, when $n > 0$. You may find it helpful to first find further derivatives beyond the fourth.
- (ii) Hence write down a formula in terms of n for $f^{(n)}(0)$, the value of the n th derivative of f at 0.
- (iii) Hence write down, using the $+\cdots+$ notation, a formula for the Taylor polynomial of degree n about 0 for f .

Exercise 11

- Find the cubic Taylor polynomial about e for the function $f(x) = \ln x$.
- Find the quintic Taylor polynomial about π for the function $f(x) = \cos x$.

Exercise 12

Given that, for each positive integer n , the Taylor polynomial of degree n about 0 for the function $f(x) = \ln(1-x)$ is

$$p_n(x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots - \frac{1}{n}x^n,$$

find the likely value of $\ln(0.95)$ to four decimal places.

Exercise 13

The Taylor polynomials of even degree about 0 for the function $f(x) = \cos(2x)$ are

$$p_0(x) = 1, \quad p_2(x) = 1 - \frac{1}{2!}(2x)^2,$$

$$p_4(x) = 1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4,$$

$$p_6(x) = 1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6,$$

and so on.

Use these Taylor polynomials to find the likely value of $\cos(0.2)$ to four decimal places.

3 Taylor series

Exercise 14

Find the first four terms of the Taylor series about 0 for the function $f(x) = \sqrt{x+4}$.

Exercise 15

- Find the Taylor series about 1 for the function $f(x) = 1/x$, writing down enough terms to make the general pattern clear.
- Explain the pattern in words.
- Show that the series can be simplified to give

$$f(x) = 1 + (1-x) + (1-x)^2 + (1-x)^3 + \cdots$$

Exercise 16

Write down the first four terms of each of the following sums and hence identify the function whose standard Taylor series about 0 the sum represents.

$$(a) \sum_{n=0}^{\infty} x^n \quad (b) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

Exercise 17

Use the binomial series to find the Taylor series about 0 for each of the following functions. (Write down enough terms to make the general pattern clear.) In each case, state an interval of validity for the series.

(a) $f(x) = (1+x)^3$ (b) $f(x) = \frac{1}{(1+x)^4}$

(c) $f(x) = (1+x)^{-1/4}$

Exercise 18

Use your answer to Exercise 17(c) to find the likely value of $0.9^{-1/4}$ to two decimal places.

4 Manipulating Taylor series

Exercise 19

By substituting for the variable in a standard Taylor series, find the Taylor series about 0 for each of the following functions. In each case, determine an interval of validity for the series.

(a) $f(x) = \frac{1}{1-3x}$ (b) $f(x) = \sin\left(\frac{1}{3}x\right)$

(c) $f(x) = \cos(x^3)$ (d) $f(x) = e^{-x}$

Exercise 20

By rearranging as necessary, use the binomial series to find the Taylor series about 0 for each of the following functions. In each case, determine an interval of validity for the series.

(a) $f(x) = \frac{1}{(1+2x)^2}$

(b) $f(x) = \sqrt{1 - \frac{1}{3}x}$

(c) $f(x) = (2+x)^4$ (d) $f(x) = \frac{1}{(2-x)^3}$

Exercise 21

- (a) Use the binomial series to find the Taylor series about 0 for the function $f(x) = (1+x)^{1/2}$.
- (b) Use your answer to part (a) to find the Taylor series about 1 for the function $f(x) = x^{1/2}$. Determine an interval of validity for this series.
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Exercise 22

Replace each occurrence of x by $x + \pi/3$ in the Taylor series about 0 for the cosine function. For what function is the resulting series the Taylor series, and what is the centre of this Taylor series?

Exercise 23

- (a) Sketch the graph of each of the following functions over at least the interval from $-\pi$ to π . Use your graph to determine whether the function is an odd function, an even function or neither an odd nor even function.
- (i) $f(x) = \cos\left(\frac{1}{2}x\right)$ (ii) $f(x) = \frac{1}{2}\sin x$
- (b) Use the standard Taylor series about 0 for $\cos x$ and $\sin x$ to find the quartic Taylor polynomial about 0 for each of the functions in part (a), and check that it is as you would expect from your answers to part (a).
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Exercise 24

- (a) Show that $\ln(2+x) = \ln 2 + \ln\left(1 + \frac{1}{2}x\right)$.
- (b) By using your answer to part (a) and substituting for the variable in a standard Taylor series, find the Taylor series about 0 for the function $f(x) = \ln(2+x)$. Determine an interval of validity for this series.
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Exercise 25

By rearranging as necessary, and using the standard Taylor series about 0 for $1/(1-x)$, find the Taylor series about 0 for each of the following functions. In each case, determine an interval of validity for the series.

$$(a) f(x) = \frac{2}{2-x} \quad (b) f(x) = \frac{16}{2+x}$$

Exercise 26

- (a) The Taylor series about 0 for $\ln((1+x)/(1-x))$ is

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots$$

Use this series to find the Taylor series about 0 for the function

$$f(x) = \ln\left(\frac{1-x}{1+x}\right).$$

- (b) Use the series in part (a) to find an expression for

$$\ln\left(\frac{1+x}{1-x}\right) + \ln\left(\frac{1-x}{1+x}\right).$$

Justify your answer using logarithm laws.

Exercise 27

Find the Taylor series about 0 for each of the following functions, giving the first four non-zero terms. In each case, state an interval of validity for the series.

- (a) $e^x + \cos x$ (b) $(1-x)\sin x$
(c) $(1+x)^2 e^x$

Exercise 28

- (a) Use the standard Taylor series about 0 for e^x to find the Taylor series about 0 for e^{2x} .
(b) Use the fact that $e^{2x} = e^x \times e^x$ and multiplication of the standard Taylor series about 0 for e^x to confirm the first four terms of your answer to part (a).

Exercise 29

- (a) Use the standard Taylor series about 0 for $\cos x$ to find the first three terms of the Taylor series about 0 for the function $f(x) = \cos^2 x$.
(b) Use your result from part (a) to find the first three terms of the Taylor series about 0 for the function $f(x) = 2\cos^2 x - 1$.
(c) Show that your answer to part (b) is consistent with the Taylor series about 0 for $\cos(2x)$.

Exercise 30

- (a) By differentiating the standard Taylor series about 0 for $\ln(1+x)$, find the Taylor series about 0 for the function

$$f(x) = \frac{1}{1+x}.$$

- (b) Use your answer to part (a) to find the Taylor series about 0 for the function

$$f(x) = \frac{1}{1+4x^2}.$$

Determine an interval of validity for this series.

Exercise 31

- (a) Find the Taylor series about 0 for the function $f(x) = (1+x)^{-1/2}$.
(b) Integrate your answer to part (a), and compare it with your answer to Exercise 21(a).

Exercise 32

The binomial series can be used to show that the Taylor series about 0 for $1/(1+x)^2$ is

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots,$$

for $-1 < x < 1$.

- (a) Use differentiation to find the Taylor series about 0 for the function

$$f(x) = \frac{1}{(1+x)^3}.$$

- (b) Use integration to find the Taylor series about 0 for the function

$$f(x) = \frac{1}{(1+x)}.$$

Solutions to exercises

Solution to Exercise 1

- (a) Since $f(0) = e^{-0} = e^0 = 1$, the constant Taylor polynomial about 0 for $f(x) = e^{-x}$ is $p(x) = 1$.

- (b) The approximation for $e^{-0.1}$ given by p is

$$p(0.1) = 1.$$

The corresponding remainder is

$$\begin{aligned} f(0.1) - p(0.1) &= e^{-0.1} - 1 \\ &= -0.09516 \quad (\text{to 5 d.p.}). \end{aligned}$$

Similarly, the approximation for $e^{-0.01}$ given by p is

$$p(0.01) = 1.$$

The corresponding remainder is

$$\begin{aligned} f(0.01) - p(0.01) &= e^{-0.01} - 1 \\ &= -0.00995 \quad (\text{to 5 d.p.}). \end{aligned}$$

Solution to Exercise 2

- (a) Since $f(1) = 1/(1+1) = \frac{1}{2}$, the constant Taylor polynomial about 1 for $f(x) = 1/(1+x)$ is $p(x) = \frac{1}{2}$.

- (b) We have

$$\frac{1}{2.1} = \frac{1}{1+1.1} = f(1.1).$$

Therefore the approximation for $\frac{1}{2.1}$ given by p is

$$p(1.1) = \frac{1}{2}.$$

The corresponding remainder is

$$\begin{aligned} f(1.1) - p(1.1) &= \frac{1}{2.1} - \frac{1}{2} \\ &= -0.02381 \quad (\text{to 5 d.p.}). \end{aligned}$$

Similarly, we have

$$\frac{1}{1.9} = \frac{1}{1+0.9} = f(0.9).$$

Therefore the approximation for $\frac{1}{1.9}$ given by p is

$$p(0.9) = \frac{1}{2}.$$

The corresponding remainder is

$$\begin{aligned} f(0.9) - p(0.9) &= \frac{1}{1.9} - \frac{1}{2} \\ &= 0.02632 \quad (\text{to 5 d.p.}). \end{aligned}$$

Solution to Exercise 3

- (a) We have $f(x) = e^{-x}$, so, by the chain rule,

$$f'(x) = -e^{-x}.$$

Hence

$$f(0) = e^0 = 1$$

and

$$f'(0) = -e^0 = -1.$$

Therefore the linear Taylor polynomial about 0 for $f(x) = e^{-x}$ is

$$p(x) = f(0) + f'(0)x = 1 - x.$$

- (b) The approximation for $e^{-0.1}$ given by the linear Taylor polynomial p is

$$p(0.1) = 1 - 0.1 = 0.9.$$

The corresponding remainder is

$$\begin{aligned} f(0.1) - p(0.1) &= e^{-0.1} - 0.9 \\ &= 0.00484 \quad (\text{to 5 d.p.}). \end{aligned}$$

Similarly, the approximation for $e^{-0.01}$ given by the linear Taylor polynomial p is

$$p(0.01) = 1 - 0.01 = 0.99.$$

The corresponding remainder is

$$\begin{aligned} f(0.01) - p(0.01) &= e^{-0.01} - 0.99 \\ &= 0.00005 \quad (\text{to 5 d.p.}). \end{aligned}$$

- (c) The remainders found here have a much smaller magnitude than the remainders found in Exercise 1(b). The linear Taylor polynomial about 0 for $f(x) = e^{-x}$ provides a much better approximation for the values of $e^{-0.1}$ and $e^{-0.01}$ than the constant Taylor polynomial.

Solution to Exercise 4

- (a) We have $f(x) = (4-x)^{1/2}$, so

$$\begin{aligned} f'(x) &= \frac{1}{2}(4-x)^{-1/2} \times (-1) \\ &= -\frac{1}{2(4-x)^{1/2}}. \end{aligned}$$

Hence

$$f(0) = (4-0)^{1/2} = 2$$

and

$$f'(0) = -\frac{1}{2(4-0)^{1/2}} = -\frac{1}{4}.$$

Therefore the linear Taylor polynomial about 0 for $f(x) = (4 - x)^{1/2}$ is

$$\begin{aligned} p(x) &= f(0) + f'(0)x \\ &= 2 - \frac{1}{4}x. \end{aligned}$$

(b) We have

$$3.98^{1/2} = (4 - 0.02)^{1/2} = f(0.02).$$

Therefore the approximation for $3.98^{1/2}$ given by the linear Taylor polynomial p is

$$p(0.02) = 2 - \frac{1}{4} \times 0.02 = 1.995.$$

To eight decimal places, the remainder is

$$\begin{aligned} f(0.02) - p(0.02) &= 3.98^{1/2} - 1.995 \\ &= -0.000\,006\,27. \end{aligned}$$

Solution to Exercise 5

We have $f(x) = e^{-x}$, so

$$f'(x) = -e^{-x}.$$

Hence

$$f(1) = e^{-1} = \frac{1}{e}$$

and

$$f'(1) = -e^{-1} = -\frac{1}{e}.$$

Therefore the linear Taylor polynomial about 1 for $f(x) = e^{-x}$ is

$$\begin{aligned} p(x) &= f(1) + f'(1)(x - 1) \\ &= \frac{1}{e} - \frac{1}{e}(x - 1) \\ &= \frac{2}{e} - \frac{1}{e}x. \end{aligned}$$

Solution to Exercise 6

We have $f(x) = 1/(1 + x) = (1 + x)^{-1}$, so

$$f'(x) = -(1 + x)^{-2} = -\frac{1}{(1 + x)^2}.$$

Hence

$$f(2) = \frac{1}{1 + 2} = \frac{1}{3}$$

and

$$f'(2) = -\frac{1}{(1 + 2)^2} = -\frac{1}{9}.$$

Therefore the linear Taylor polynomial about 2 for $f(x) = 1/(1 + x)$ is

$$\begin{aligned} p(x) &= f(2) + f'(2)(x - 2) \\ &= \frac{1}{3} - \frac{1}{9}(x - 2) \\ &= \frac{5}{9} - \frac{1}{9}x. \end{aligned}$$

Solution to Exercise 7

(a) We have $f(x) = e^{2x}$, so

$$f'(x) = 2e^{2x} \quad \text{and} \quad f''(x) = 4e^{2x}.$$

Hence

$$f(0) = e^0 = 1, \quad f'(0) = 2e^0 = 2$$

and

$$f''(0) = 4e^0 = 4.$$

Therefore the quadratic Taylor polynomial about 0 for $f(x) = e^{2x}$ is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 \\ &= 1 + 2x + 2x^2. \end{aligned}$$

(b) We have $f(x) = \sin(\frac{1}{2}x)$, so

$$f'(x) = \frac{1}{2} \cos(\frac{1}{2}x)$$

and

$$f''(x) = -\frac{1}{4} \sin(\frac{1}{2}x).$$

Hence

$$f(0) = \sin 0 = 0, \quad f'(0) = \frac{1}{2} \cos 0 = \frac{1}{2}$$

and

$$f''(0) = -\frac{1}{4} \sin 0 = 0.$$

Therefore the quadratic Taylor polynomial about 0 for $f(x) = \sin(\frac{1}{2}x)$ is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 \\ &= 0 + \frac{1}{2}x + 0 \\ &= \frac{1}{2}x. \end{aligned}$$

Solution to Exercise 8

(a) We have $f(x) = x \cos x$, so, by the product rule,

$$f'(x) = \cos x - x \sin x$$

and

$$\begin{aligned} f''(x) &= -\sin x - (\sin x + x \cos x) \\ &= -2 \sin x - x \cos x. \end{aligned}$$

Hence

$$f(0) = 0 \times \cos 0 = 0,$$

$$f'(0) = \cos 0 - 0 \times \sin 0 = 1$$

and

$$f''(0) = -2 \times \sin 0 - 0 \times \cos 0 = 0.$$

Therefore the quadratic Taylor polynomial about 0 for $f(x) = x \cos x$ is

$$p(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = x.$$

- (b) The approximation for $f(0.01)$ given by the quadratic Taylor polynomial p is

$$p(0.01) = 0.01.$$

To eight decimal places, the remainder is

$$f(0.01) - p(0.01) = 0.01 \cos(0.01) - 0.01 = -0.000\,000\,50.$$

Solution to Exercise 9

- (a) We have $f(x) = e^{2x}$, so

$$f'(x) = 2e^{2x} \quad \text{and} \quad f''(x) = 4e^{2x}.$$

Hence

$$f(1) = e^2, \quad f'(1) = 2e^2$$

and

$$f''(1) = 4e^2.$$

Therefore the quadratic Taylor polynomial about 1 for $f(x) = e^{2x}$ is

$$p(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = e^2 + 2e^2(x-1) + 2e^2(x-1)^2.$$

- (b) We have $f(x) = x^{1/2}$, so

$$f'(x) = \frac{1}{2x^{1/2}} \quad \text{and} \quad f''(x) = -\frac{1}{4x^{3/2}}.$$

Hence

$$f(1) = 1, \quad f'(1) = \frac{1}{2} \quad \text{and} \quad f''(1) = -\frac{1}{4}.$$

Therefore the quadratic Taylor polynomial about 1 for $f(x) = x^{1/2}$ is

$$p(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2.$$

Solution to Exercise 10

- (a) To find the quartic Taylor polynomial about 0 for any function $f(x)$ we need to evaluate $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$ and $f^{(4)}(0)$.

- (i) For $f(x) = \cos(2x)$, we have:

$$\begin{aligned} f(x) &= \cos(2x), & f(0) &= 1; \\ f'(x) &= -2\sin(2x), & f'(0) &= 0; \\ f''(x) &= -4\cos(2x), & f''(0) &= -4; \\ f^{(3)}(x) &= 8\sin(2x), & f^{(3)}(0) &= 0; \\ f^{(4)}(x) &= 16\cos(2x), & f^{(4)}(0) &= 16. \end{aligned}$$

Therefore the quartic Taylor polynomial about 0 for $f(x) = \cos(2x)$ is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\ &\quad + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 - \frac{4}{2!}x^2 + \frac{16}{4!}x^4 \\ &= 1 - 2x^2 + \frac{2}{3}x^4. \end{aligned}$$

- (ii) To make the differentiation easier, note that

$$f(x) = \ln\left(\frac{1}{1+x}\right) = -\ln(1+x).$$

Thus we have:

$$\begin{aligned} f(x) &= -\ln(1+x), & f(0) &= 0; \\ f'(x) &= -\frac{1}{1+x}, & f'(0) &= -1; \\ f''(x) &= \frac{1}{(1+x)^2}, & f''(0) &= 1; \\ f^{(3)}(x) &= -\frac{2}{(1+x)^3}, & f^{(3)}(0) &= -2; \\ f^{(4)}(x) &= \frac{6}{(1+x)^4}, & f^{(4)}(0) &= 6. \end{aligned}$$

Therefore the quartic Taylor polynomial about 0 for $f(x) = \ln(1/(1+x))$ is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\ &\quad + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= -x + \frac{1}{2!}x^2 - \frac{2}{3!}x^3 + \frac{6}{4!}x^4 \\ &= -x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4. \end{aligned}$$

- (b) (i) The first four derivatives of $f(x) = \ln(1/(1+x))$ found in part (a)(ii) can be written as

$$\begin{aligned} f'(x) &= -\frac{0!}{1+x}, \\ f''(x) &= \frac{1!}{(1+x)^2}, \\ f^{(3)}(x) &= -\frac{2!}{(1+x)^3}, \\ f^{(4)}(x) &= \frac{3!}{(1+x)^4}. \end{aligned}$$

Exercise Booklet 11

The n th derivative of f will be

$$f^{(n)}(x) = (-1)^n \frac{(n-1)!}{(1+x)^n}, \text{ for } n > 0.$$

(The expression $(-1)^n$ produces the alternating signs.)

- (ii) By putting $x = 0$ in the formula for $f^{(n)}(x)$, we obtain

$$\begin{aligned} f^{(n)}(0) &= (-1)^n \frac{(n-1)!}{(1+0)^n} \\ &= (-1)^n (n-1)!. \end{aligned}$$

- (iii) Hence the Taylor polynomial of degree n about 0 for $f(x) = \ln(1/(1+x))$ is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\ &\quad + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= -x + \frac{1}{2!}x^2 - \frac{2}{3!}x^3 \\ &\quad + \cdots + \frac{(-1)^n (n-1)!}{n!}x^n \\ &= -x + \frac{1}{2}x^2 - \frac{1}{3}x^3 \\ &\quad + \cdots + \frac{(-1)^n}{n}x^n. \end{aligned}$$

Solution to Exercise 11

- (a) The cubic Taylor polynomial about e for a function f is given by

$$\begin{aligned} p(x) &= f(e) + f'(e)(x-e) + \frac{f''(e)}{2!}(x-e)^2 \\ &\quad + \frac{f^{(3)}(e)}{3!}(x-e)^3. \end{aligned}$$

We have:

$$\begin{aligned} f(x) &= \ln x, & f(e) &= \ln e = 1; \\ f'(x) &= \frac{1}{x}, & f'(e) &= \frac{1}{e}; \\ f''(x) &= -\frac{1}{x^2}, & f''(e) &= -\frac{1}{e^2}; \\ f^{(3)}(x) &= \frac{2}{x^3}, & f^{(3)}(e) &= \frac{2}{e^3}. \end{aligned}$$

Therefore the cubic Taylor polynomial about e for $\ln x$ is

$$\begin{aligned} p(x) &= 1 + \frac{1}{e}(x-e) - \frac{1}{2!e^2}(x-e)^2 \\ &\quad + \frac{2}{3!e^3}(x-e)^3 \\ &= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 \\ &\quad + \frac{1}{3e^3}(x-e)^3. \end{aligned}$$

- (b) The quintic Taylor polynomial about π for a function f is given by

$$\begin{aligned} p(x) &= f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 \\ &\quad + \cdots + \frac{f^{(5)}(\pi)}{5!}(x-\pi)^5. \end{aligned}$$

We have:

$$\begin{aligned} f(x) &= \cos x, & f(\pi) &= -1; \\ f'(x) &= -\sin x, & f'(\pi) &= 0; \\ f''(x) &= -\cos x, & f''(\pi) &= 1; \\ f^{(3)}(x) &= \sin x, & f^{(3)}(\pi) &= 0; \\ f^{(4)}(x) &= \cos x, & f^{(4)}(\pi) &= -1; \\ f^{(5)}(x) &= -\sin x, & f^{(5)}(\pi) &= 0. \end{aligned}$$

Therefore the quintic Taylor polynomial about π for $\cos x$ is

$$\begin{aligned} p(x) &= -1 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 \\ &= -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4. \end{aligned}$$

Solution to Exercise 12

We have $\ln(0.95) = \ln(1 - 0.05) = f(0.05)$.

Using the given Taylor polynomials

$$p_n(x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots - \frac{1}{n}x^n,$$

and calculating values to six decimal places, we obtain

$$\begin{aligned} p_1(0.05) &= -0.05 \\ p_2(0.05) &= p_1(0.05) - \frac{1}{2}(0.05)^2 \\ &= -0.05125 \\ p_3(0.05) &= p_2(0.05) - \frac{1}{3}(0.05)^3 \\ &= -0.051292 \quad (\text{to 6 d.p.}) \\ p_4(0.05) &= p_3(0.05) - \frac{1}{4}(0.05)^4 \\ &= -0.051293 \quad (\text{to 6 d.p.}) \\ p_5(0.05) &= p_4(0.05) - \frac{1}{5}(0.05)^5 \\ &= -0.051293 \quad (\text{to 6 d.p.}). \end{aligned}$$

The values of $p_4(0.05)$ and $p_5(0.05)$ agree to six decimal places, so it is likely that

$$\ln(0.95) = -0.0513$$

to four decimal places. (A calculator shows that this is indeed the case.)

Solution to Exercise 13

We have $\cos(0.2) = \cos(2 \times 0.1) = f(0.1)$. Using the given Taylor polynomials, and calculating values to six decimal places, we obtain

$$p_0(0.1) = 1$$

$$\begin{aligned} p_2(0.1) &= p_0(0.1) - \frac{1}{2!}(2 \times 0.1)^2 \\ &= 0.98 \end{aligned}$$

$$\begin{aligned} p_4(0.1) &= p_2(0.1) + \frac{1}{4!}(2 \times 0.1)^4 \\ &= 0.980067 \quad (\text{to 6 d.p.}) \end{aligned}$$

$$\begin{aligned} p_6(0.1) &= p_4(0.1) - \frac{1}{6!}(2 \times 0.1)^6 \\ &= 0.980067 \quad (\text{to 6 d.p.}). \end{aligned}$$

The values of $p_4(0.1)$ and $p_6(0.1)$ agree to six decimal places, so it is likely that

$$\cos(0.2) = 0.9801$$

to four decimal places. (A calculator shows that this is indeed the case.)

Solution to Exercise 14

To find the first four terms of the Taylor series about 0 for $f(x) = \sqrt{x+4} = (x+4)^{1/2}$ we need to evaluate $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$. We have:

$$\begin{aligned} f(x) &= (x+4)^{1/2}, & f(0) &= 4^{1/2} \\ & & &= 2; \end{aligned}$$

$$\begin{aligned} f'(x) &= \frac{1}{2(x+4)^{1/2}}, & f'(0) &= \frac{1}{2 \times 4^{1/2}} \\ & & &= \frac{1}{4}; \end{aligned}$$

$$\begin{aligned} f''(x) &= -\frac{1}{4(x+4)^{3/2}}, & f''(0) &= -\frac{1}{4 \times 4^{3/2}} \\ & & &= -\frac{1}{32}; \end{aligned}$$

$$\begin{aligned} f^{(3)}(x) &= \frac{3}{8(x+4)^{5/2}}, & f^{(3)}(0) &= \frac{3}{8 \times 4^{5/2}} \\ & & &= \frac{3}{256}. \end{aligned}$$

Therefore the Taylor series about 0 for $f(x) = \sqrt{x+4}$ is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots \\ = 2 + \frac{1}{4}x - \frac{\frac{1}{32}}{2!}x^2 + \frac{\frac{3}{256}}{3!}x^3 + \dots \\ = 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3 + \dots \end{aligned}$$

Solution to Exercise 15

- (a) To find the Taylor series about 1 for $f(x) = 1/x$, we need to evaluate $f(1)$, $f'(1)$, $f''(1)$, $f^{(3)}(1)$, $\dots$. We have:

$$f(x) = \frac{1}{x}, \quad f(1) = 1;$$

$$f'(x) = -\frac{1}{x^2}, \quad f'(1) = -1;$$

$$f''(x) = \frac{2}{x^3}, \quad f''(1) = 2;$$

$$f^{(3)}(x) = -\frac{6}{x^4}, \quad f^{(3)}(1) = -6.$$

Therefore the Taylor series about 1 for $f(x) = 1/x$ is

$$\begin{aligned} f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \dots \\ = 1 - (x-1) + \frac{2}{2!}(x-1)^2 \\ - \frac{6}{3!}(x-1)^3 + \dots \\ = 1 - (x-1) + (x-1)^2 \\ - (x-1)^3 + \dots \end{aligned}$$

- (b) The terms in the Taylor series are terms in increasing powers of $x-1$. When the power of $x-1$ is odd, the sign of the term is negative and when the power of $x-1$ is even, the sign of the term is positive.
- (c) By taking the appropriate power of -1 from each bracket this Taylor series can be rewritten as

$$\begin{aligned} f(x) &= 1 - (-1)(1-x) + (-1)^2(1-x)^2 \\ &\quad - (-1)^3(1-x)^3 + \dots \\ &= 1 + (1-x) + (1-x)^2 \\ &\quad + (1-x)^3 + \dots \end{aligned}$$

Solution to Exercise 16

$$\begin{aligned}
 \text{(a)} \quad \sum_{n=0}^{\infty} x^n &= x^0 + x^1 + x^2 + x^3 + \dots \\
 &= 1 + x + x^2 + x^3 + \dots \\
 &= \frac{1}{1-x}, \quad \text{for } -1 < x < 1. \\
 \text{(b)} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n &= \frac{(-1)^0}{1} x^1 + \frac{(-1)^1}{2} x^2 \\
 &\quad + \frac{(-1)^2}{3} x^3 + \frac{(-1)^3}{4} x^4 + \dots \\
 &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \\
 &= \ln(1+x), \\
 &\quad \text{for } -1 < x < 1.
 \end{aligned}$$

Solution to Exercise 17

$$\begin{aligned}
 \text{(a)} \quad \text{Taking } \alpha = 3 \text{ in the binomial series gives} \\
 (1+x)^3 &= 1 + 3x + \frac{3 \times 2}{2!} x^2 + \frac{3 \times 2 \times 1}{3!} x^3 \\
 &\quad + \frac{3 \times 2 \times 1 \times 0}{4!} x^4 \\
 &\quad + \frac{3 \times 2 \times 1 \times 0 \times (-1)}{5!} x^5 + \dots \\
 &= 1 + 3x + 3x^2 + x^3.
 \end{aligned}$$

This Taylor series is valid for $x \in \mathbb{R}$.

- (b) Since $1/(1+x)^4 = (1+x)^{-4}$, we take $\alpha = -4$ in the binomial series, to give

$$\begin{aligned}
 \frac{1}{(1+x)^4} &= 1 + (-4)x + \frac{(-4)(-5)}{2!} x^2 \\
 &\quad + \frac{(-4)(-5)(-6)}{3!} x^3 \\
 &\quad + \frac{(-4)(-5)(-6)(-7)}{4!} x^4 + \dots \\
 &= 1 - 4x + 10x^2 - 20x^3 \\
 &\quad + 35x^4 - \dots.
 \end{aligned}$$

This Taylor series is valid for $-1 < x < 1$.

- (c) Taking $\alpha = -\frac{1}{4}$ in the binomial series gives

$$\begin{aligned}
 (1+x)^{-1/4} &= 1 + \left(-\frac{1}{4}\right)x + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!} x^2 \\
 &\quad + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!} x^3 \\
 &\quad + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)\left(-\frac{13}{4}\right)}{4!} x^4 \\
 &\quad + \dots \\
 &= 1 - \frac{1}{4}x + \frac{5}{4^2 \times 2!} x^2 \\
 &\quad - \frac{5 \times 9}{4^3 \times 3!} x^3 \\
 &\quad + \frac{5 \times 9 \times 13}{4^4 \times 4!} x^4 - \dots.
 \end{aligned}$$

This Taylor series is valid for $-1 < x < 1$.

(The coefficients of the terms of the Taylor series here have been left in a simplified but unevaluated form, to make the general pattern clear.)

Solution to Exercise 18

We have $0.9^{-1/4} = (1 + (-0.1))^{-1/4}$. Using the series from the solution to Exercise 17(c), and calculating values to four decimal places, we obtain

$$\begin{aligned}
 p_1(-0.1) &= 1 - \frac{1}{4}(-0.1) \\
 &= 1.025 \\
 p_2(-0.1) &= p_1(-0.1) + \frac{5}{4^2 \times 2!} (-0.1)^2 \\
 &= 1.0266 \quad (\text{to 4 d.p.}) \\
 p_3(-0.1) &= p_2(-0.1) - \frac{5 \times 9}{4^3 \times 3!} (-0.1)^3 \\
 &= 1.0267 \quad (\text{to 4 d.p.}) \\
 p_4(-0.1) &= p_3(-0.1) + \frac{5 \times 9 \times 13}{4^4 \times 4!} (-0.1)^4 \\
 &= 1.0267 \quad (\text{to 4 d.p.}).
 \end{aligned}$$

The values of $p_3(-0.1)$ and $p_4(-0.1)$ agree to four decimal places, so it is likely that

$$0.9^{-1/4} = 1.03$$

to two decimal places. (A calculator shows that this is indeed the case.)

Solution to Exercise 19

- (a) The Taylor series about 0 for
- $1/(1-x)$
- is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots,$$

for $-1 < x < 1$. Replacing each occurrence of x by $3x$ gives

$$\begin{aligned}\frac{1}{1-3x} &= 1 + (3x) + (3x)^2 + (3x)^3 + \dots \\ &= 1 + 3x + 9x^2 + 27x^3 + \dots\end{aligned}$$

This is the Taylor series about 0 for $1/(1-3x)$. It is valid for $-1 < 3x < 1$; that is, for $-\frac{1}{3} < x < \frac{1}{3}$.

- (b) The Taylor series about 0 for
- $\sin x$
- is

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots,$$

for $x \in \mathbb{R}$. Replacing each occurrence of x by $\frac{1}{3}x$ gives

$$\begin{aligned}\sin\left(\frac{1}{3}x\right) &= \left(\frac{1}{3}x\right) - \frac{1}{3!}\left(\frac{1}{3}x\right)^3 + \frac{1}{5!}\left(\frac{1}{3}x\right)^5 \\ &\quad - \frac{1}{7!}\left(\frac{1}{3}x\right)^7 + \dots \\ &= \frac{1}{3}x - \frac{1}{3^3 \times 3!}x^3 + \frac{1}{3^5 \times 5!}x^5 \\ &\quad - \frac{1}{3^7 \times 7!}x^7 + \dots\end{aligned}$$

This is the Taylor series about 0 for $\sin(\frac{1}{3}x)$. It is valid for $x \in \mathbb{R}$.

- (c) The Taylor series about 0 for
- $\cos x$
- is

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots,$$

for $x \in \mathbb{R}$. Replacing each occurrence of x by x^3 gives

$$\begin{aligned}\cos(x^3) &= 1 - \frac{1}{2!}(x^3)^2 + \frac{1}{4!}(x^3)^4 \\ &\quad - \frac{1}{6!}(x^3)^6 + \dots \\ &= 1 - \frac{1}{2!}x^6 + \frac{1}{4!}x^{12} \\ &\quad - \frac{1}{6!}x^{18} + \dots\end{aligned}$$

This is the Taylor series about 0 for $\cos(x^3)$. It is valid for $x \in \mathbb{R}$.

- (d) The Taylor series about 0 for
- e^x
- is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots,$$

for $x \in \mathbb{R}$. Replacing each occurrence of x by $-x$ gives

$$\begin{aligned}e^{-x} &= 1 + (-x) + \frac{1}{2!}(-x)^2 + \frac{1}{3!}(-x)^3 \\ &\quad + \frac{1}{4!}(-x)^4 + \dots \\ &= 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \dots\end{aligned}$$

This is the Taylor series about 0 for e^{-x} . It is valid for $x \in \mathbb{R}$.

Solution to Exercise 20

- (a) Since
- $1/(1+2x)^2 = (1+2x)^{-2}$
- , we take
- $\alpha = -2$
- and replace each occurrence of
- x
- by
- $2x$
- in the binomial series, to give

$$\begin{aligned}\frac{1}{(1+2x)^2} &= 1 + (-2)(2x) + \frac{(-2)(-3)}{2!}(2x)^2 \\ &\quad + \frac{(-2)(-3)(-4)}{3!}(2x)^3 + \dots \\ &= 1 - 4x + 12x^2 - 32x^3 + \dots\end{aligned}$$

This Taylor series is valid for $-1 < 2x < 1$; that is, for $-\frac{1}{2} < x < \frac{1}{2}$.

- (b) Since
- $\sqrt{1-\frac{1}{3}x} = (1+(-\frac{1}{3}x))^{1/2}$
- , we take
- $\alpha = \frac{1}{2}$
- and replace each occurrence of
- x
- by
- $-\frac{1}{3}x$
- in the binomial series, to give

$$\begin{aligned}\sqrt{1-\frac{1}{3}x} &= 1 + \frac{1}{2}\left(-\frac{1}{3}x\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(-\frac{1}{3}x\right)^2 \\ &\quad + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(-\frac{1}{3}x\right)^3 + \dots \\ &= 1 - \frac{1}{6}x - \frac{1}{72}x^2 - \frac{1}{432}x^3 - \dots\end{aligned}$$

This Taylor series is valid for $-1 < -\frac{1}{3}x < 1$; that is, for $-3 < -x < 3$, or equivalently for $-3 < x < 3$.

- (c) Since
- $(2+x)^4 = 2^4(1+\frac{1}{2}x)^4$
- , we take
- $\alpha = 4$
- and replace each occurrence of
- x
- by
- $\frac{1}{2}x$
- in the binomial series, to give

$$\begin{aligned}(2+x)^4 &= 2^4 \left(1 + 4\left(\frac{1}{2}x\right) + \frac{4 \times 3}{2!}\left(\frac{1}{2}x\right)^2 \right. \\ &\quad + \frac{4 \times 3 \times 2}{3!}\left(\frac{1}{2}x\right)^3 \\ &\quad + \frac{4 \times 3 \times 2 \times 1}{4!}\left(\frac{1}{2}x\right)^4 \\ &\quad \left. + \frac{4 \times 3 \times 2 \times 1 \times 0}{5!}\left(\frac{1}{2}x\right)^5 + \dots \right) \\ &= 2^4 \left(1 + 2x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{16}x^4 \right) \\ &= 16 + 32x + 24x^2 + 8x^3 + x^4.\end{aligned}$$

This Taylor series is valid for $x \in \mathbb{R}$.

(A simpler way to expand $(2+x)^4$ is to use the binomial theorem.)

(d) Since

$$f(x) = \frac{1}{(2-x)^3} = \frac{1}{2^3} \times \left(1 - \frac{1}{2}x\right)^{-3},$$

we take $\alpha = -3$ and replace each occurrence of x by $-\frac{1}{2}x$ in the binomial series to give

$$\begin{aligned} \frac{1}{(2-x)^3} &= \frac{1}{2^3} \left(1 + (-3)\left(-\frac{1}{2}x\right) \right. \\ &\quad + \frac{(-3)(-4)}{2!} \left(-\frac{1}{2}x\right)^2 \\ &\quad \left. + \frac{(-3)(-4)(-5)}{3!} \left(-\frac{1}{2}x\right)^3 + \dots \right) \\ &= \frac{1}{2^3} \left(1 + \frac{3}{2}x + \frac{3}{2}x^2 + \frac{5}{4}x^3 + \dots \right) \\ &= \frac{1}{2^3} + \frac{3}{2^4}x + \frac{3}{2^4}x^2 + \frac{5}{2^5}x^3 + \dots \end{aligned}$$

This Taylor series is valid for $-1 < -\frac{1}{2}x < 1$; that is, for $-2 < -x < 2$, or equivalently for $-2 < x < 2$.

Solution to Exercise 21

(a) Taking $\alpha = \frac{1}{2}$ in the binomial series gives

$$\begin{aligned} (1+x)^{1/2} &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 \\ &\quad + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \dots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots, \end{aligned}$$

for $-1 < x < 1$.

(b) We can use the fact that

$$(1+(x-1))^{1/2} = x^{1/2},$$

so replacing each occurrence of x by $x-1$ in the Taylor series about 0 for $(1+x)^{1/2}$ gives

$$\begin{aligned} x^{1/2} &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 \\ &\quad + \frac{1}{16}(x-1)^3 - \dots \end{aligned}$$

This is the Taylor series about 1 for $f(x) = x^{1/2}$. The interval of validity is $-1 < x-1 < 1$; that is, for $0 < x < 2$.

Solution to Exercise 22

Replacing each occurrence of x by $x + \pi/3$ in the Taylor series about 0 for the cosine function gives

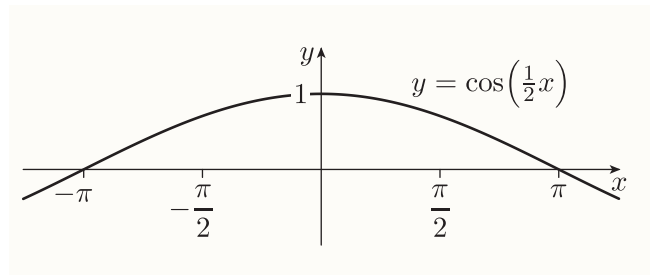
$$\begin{aligned} \cos\left(x + \frac{\pi}{3}\right) &= 1 - \frac{1}{2!}\left(x + \frac{\pi}{3}\right)^2 \\ &\quad + \frac{1}{4!}\left(x + \frac{\pi}{3}\right)^4 - \dots, \end{aligned}$$

for $x \in \mathbb{R}$.

This is the Taylor series for the function $f(x) = \cos(x + \pi/3)$, with centre $-\pi/3$.

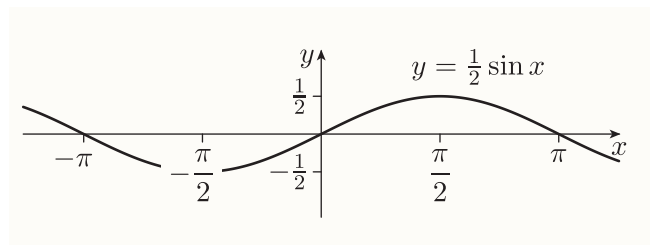
Solution to Exercise 23

(a) (i) The graph is as follows.



This graph is unchanged under reflection in the y -axis, so $f(x) = \cos\left(\frac{1}{2}x\right)$ is an even function.

(ii) The graph is as follows.



This graph is unchanged by a rotation through a half-turn about the origin, so $f(x) = \frac{1}{2} \sin x$ is an odd function.

(b) (i) The quartic Taylor polynomial about 0 for $\cos x$ is obtained by truncating the Taylor series, to give

$$p(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4.$$

Replacing each occurrence of x by $\frac{1}{2}x$ gives the quartic Taylor polynomial about 0 for $\cos\left(\frac{1}{2}x\right)$, which is

$$p\left(\frac{1}{2}x\right) = 1 - \frac{1}{8}x^2 + \frac{1}{384}x^4.$$

This polynomial contains only even powers of x , as expected for a Taylor polynomial for an even function.

(ii) The quartic Taylor polynomial about 0 for $\sin x$ is obtained by truncating the Taylor series, to give

$$p(x) = x - \frac{1}{3!}x^3 = x - \frac{1}{6}x^3.$$

The quartic Taylor polynomial about 0 for $\frac{1}{2} \sin x$ is therefore

$$\frac{1}{2}p(x) = \frac{1}{2}x - \frac{1}{12}x^3.$$

This polynomial contains only odd powers of x , as expected for a Taylor polynomial for an odd function.

Solution to Exercise 24

- (a) Factorising and using a logarithm law gives

$$\begin{aligned}\ln(2+x) &= \ln\left(2\left(1+\frac{1}{2}x\right)\right) \\ &= \ln 2 + \ln\left(1+\frac{1}{2}x\right).\end{aligned}$$

- (b) The Taylor series about 0 for $\ln(1+x)$ is

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots,$$

for $-1 < x < 1$. Replacing each occurrence of x by $\frac{1}{2}x$ gives

$$\begin{aligned}\ln\left(1+\frac{1}{2}x\right) &= \left(\frac{1}{2}x\right) - \frac{1}{2}\left(\frac{1}{2}x\right)^2 + \frac{1}{3}\left(\frac{1}{2}x\right)^3 \\ &\quad - \frac{1}{4}\left(\frac{1}{2}x\right)^4 + \dots \\ &= \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{24}x^3 \\ &\quad - \frac{1}{64}x^4 + \dots.\end{aligned}$$

The Taylor series about 0 for $\ln(2+x)$ is therefore

$$\begin{aligned}\ln(2+x) &= \ln 2 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{24}x^3 \\ &\quad - \frac{1}{64}x^4 + \dots.\end{aligned}$$

It is valid for $-1 < \frac{1}{2}x < 1$; that is, for $-2 < x < 2$.

Solution to Exercise 25

In each part we use the Taylor series about 0 for $1/(1-x)$, which is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots,$$

for $-1 < x < 1$.

- (a) We can rearrange f to give

$$f(x) = \frac{2}{2-x} = \frac{1}{1-\frac{1}{2}x}.$$

Replacing each occurrence of x by $\frac{1}{2}x$ in the Taylor series for $1/(1-x)$ gives

$$\begin{aligned}\frac{2}{2-x} &= 1 + \left(\frac{1}{2}x\right) + \left(\frac{1}{2}x\right)^2 + \left(\frac{1}{2}x\right)^3 + \dots \\ &= 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \dots.\end{aligned}$$

This is valid for $-1 < \frac{1}{2}x < 1$; that is, for $-2 < x < 2$.

- (b) We can rearrange f to give

$$f(x) = \frac{16}{2+x} = \frac{8}{1+\frac{1}{2}x} = 8 \left(\frac{1}{1-(-\frac{1}{2}x)} \right).$$

Replacing each occurrence of x by $-\frac{1}{2}x$ in the Taylor series for $1/(1-x)$ and multiplying the result by 8 gives

$$\begin{aligned}\frac{16}{2+x} &= 8 \left(1 + \left(-\frac{1}{2}x\right) + \left(-\frac{1}{2}x\right)^2 \right. \\ &\quad \left. + \left(-\frac{1}{2}x\right)^3 + \dots \right) \\ &= 8 - 4x + 2x^2 - x^3 + \dots.\end{aligned}$$

This is valid for $-1 < -\frac{1}{2}x < 1$; that is, for $-2 < -x < 2$, or equivalently for $-2 < x < 2$.

Solution to Exercise 26

- (a) Replacing each occurrence of x by $-x$ in the given series we obtain

$$\ln\left(\frac{1-x}{1+x}\right) = -2x - \frac{2}{3}x^3 - \frac{2}{5}x^5 - \dots.$$

This is the Taylor series about 0 for $\ln((1-x)/(1+x))$.

- (b) Using these series gives

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right) + \ln\left(\frac{1-x}{1+x}\right) \\ &= \left(2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \dots\right) \\ &\quad + \left(-2x - \frac{2}{3}x^3 - \frac{2}{5}x^5 - \dots\right) \\ &= 0.\end{aligned}$$

This is because

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

and

$$\ln\left(\frac{1-x}{1+x}\right) = \ln(1-x) - \ln(1+x),$$

which give

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right) + \ln\left(\frac{1-x}{1+x}\right) \\ &= \ln(1+x) - \ln(1-x) \\ &\quad + \ln(1-x) - \ln(1+x) \\ &= 0.\end{aligned}$$

Solution to Exercise 27

(a) We have

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots,$$

for $x \in \mathbb{R}$, and

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots,$$

for $x \in \mathbb{R}$. Hence

$$\begin{aligned} e^x + \cos x &= \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots\right) \\ &\quad + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots\right) \\ &= (1 + 1) + x + \left(\frac{1}{2!} - \frac{1}{2!}\right)x^2 + \frac{1}{3!}x^3 \\ &\quad + \left(\frac{1}{4!} + \frac{1}{4!}\right)x^4 + \dots \\ &= 2 + x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \dots, \end{aligned}$$

for $x \in \mathbb{R}$.

(b) We have

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots,$$

for $x \in \mathbb{R}$.

Multiplying both sides of this equation by $1 - x$ gives

$$\begin{aligned} (1 - x) \sin x &= 1 \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \right) \\ &\quad - x \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \right) \\ &= \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \right) \\ &\quad - \left(x^2 - \frac{1}{3!}x^4 + \frac{1}{5!}x^6 - \frac{1}{7!}x^8 + \dots \right) \\ &= x - x^2 - \frac{1}{3!}x^3 + \frac{1}{3!}x^4 + \dots \end{aligned}$$

for $x \in \mathbb{R}$.

(c) We have $(1 + x)^2 = 1 + 2x + x^2$, and

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots,$$

for $x \in \mathbb{R}$. Hence

$$\begin{aligned} (1 + x)^2 e^x &= (1 + 2x + x^2) \\ &\quad \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \right) \\ &= 1 \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \right) \\ &\quad + 2x \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \right) \\ &\quad + x^2 \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \right) \\ &= \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \right) \\ &\quad + \left(2x + 2x^2 + \frac{2}{2!}x^3 + \frac{2}{3!}x^4 + \dots \right) \\ &\quad + \left(x^2 + x^3 + \frac{1}{2!}x^4 + \frac{1}{3!}x^5 + \dots \right) \end{aligned}$$

$$= 1 + 3x + \frac{7}{2}x^2 + \frac{13}{6}x^3 + \dots,$$

for $x \in \mathbb{R}$.

Solution to Exercise 28

(a) The Taylor series about 0 for e^x is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots,$$

for $x \in \mathbb{R}$. Replacing each occurrence of x by $2x$ we obtain the Taylor series about 0 for e^{2x} :

$$\begin{aligned} e^{2x} &= 1 + (2x) + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 \\ &\quad + \frac{1}{4!}(2x)^4 + \dots \\ &= 1 + 2x + \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 + \frac{2^4}{4!}x^4 + \dots \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \dots, \end{aligned}$$

for $x \in \mathbb{R}$.

- (b) Using $e^{2x} = e^x \times e^x$, and ignoring all terms that lead to fourth or higher powers of x , we obtain

$$\begin{aligned}
 e^x \times e^x &= \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots\right) \\
 &\quad \times \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots\right) \\
 &= 1 \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots\right) \\
 &\quad + x \left(1 + x + \frac{1}{2!}x^2 + \dots\right) \\
 &\quad + \frac{1}{2!}x^2(1 + x + \dots) \\
 &\quad + \frac{1}{3!}x^3(1 + \dots) + \dots \\
 &= \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots\right) \\
 &\quad + \left(x + x^2 + \frac{1}{2!}x^3 + \dots\right) \\
 &\quad + \left(\frac{1}{2!}x^2 + \frac{1}{2!}x^3 + \dots\right) + \frac{1}{3!}x^3 + \dots \\
 &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots,
 \end{aligned}$$

for $x \in \mathbb{R}$. This is the Taylor series about 0 for e^{2x} , which agrees with the solution to part (a).

Solution to Exercise 29

- (a) The standard Taylor series for $\cos x$ is

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots,$$

for $x \in \mathbb{R}$. Hence, ignoring all terms that lead to fourth or higher powers of x , we obtain

$$\begin{aligned}
 \cos^2 x &= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots\right)^2 \\
 &= 1 \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots\right) \\
 &\quad - \frac{1}{2!}x^2 \left(1 - \frac{1}{2!}x^2 + \dots\right) \\
 &\quad + \frac{1}{4!}x^4(1 - \dots) - \dots \\
 &= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots\right) \\
 &\quad - \left(\frac{1}{2}x^2 - \frac{1}{4}x^4 + \dots\right) \\
 &\quad + \left(\frac{1}{24}x^4 - \dots\right) - \dots \\
 &= 1 - x^2 + \frac{1}{3}x^4 - \dots,
 \end{aligned}$$

for $x \in \mathbb{R}$.

- (b) Using the result from part (a) gives

$$\begin{aligned}
 2 \cos^2 x - 1 &= 2 \left(1 - x^2 + \frac{1}{3}x^4 - \dots\right) - 1 \\
 &= 1 - 2x^2 + \frac{2}{3}x^4 - \dots,
 \end{aligned}$$

for $x \in \mathbb{R}$.

- (c) The Taylor series for $\cos(2x)$ can be obtained by replacing each occurrence of x by $2x$ in the Taylor series about 0 for $\cos x$. This gives

$$\begin{aligned}
 \cos(2x) &= 1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 - \dots \\
 &= 1 - 2x^2 + \frac{2}{3}x^4 - \dots,
 \end{aligned}$$

for $x \in \mathbb{R}$, as required.

Solution to Exercise 30

- (a) The standard Taylor series about 0 for $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots,$$

for $-1 < x < 1$. Differentiating both sides of this equation gives

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots,$$

for $-1 < x < 1$. This is the Taylor series about 0 for $1/(1+x)$.

(A simpler way to obtain this Taylor series is to replace x by $-x$ in the Taylor series for $1/(1-x)$.)

- (b) Replacing each occurrence of x by $4x^2$ in the series for $1/(1+x)$ gives

$$\begin{aligned}
 \frac{1}{1+4x^2} &= 1 - (4x^2) + (4x^2)^2 - (4x^2)^3 + \dots \\
 &= 1 - 4x^2 + 16x^4 - 64x^6 + \dots.
 \end{aligned}$$

This is the Taylor series about 0 for $1/(1+4x^2)$. It is valid for $-1 < 4x^2 < 1$.

The left-hand inequality is $-1 < 4x^2$, which is equivalent to $4x^2 > -1$. This is true for all real x .

The right-hand inequality is $4x^2 < 1$; that is, $-1 < 2x < 1$ or equivalently, $-\frac{1}{2} < x < \frac{1}{2}$.

Thus this Taylor series is valid for $-\frac{1}{2} < x < \frac{1}{2}$.

Solution to Exercise 31

- (a) Taking $\alpha = -\frac{1}{2}$ in the binomial series gives

$$\begin{aligned}(1+x)^{-1/2} &= 1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 \\ &\quad + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3 + \dots \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots,\end{aligned}$$

for $-1 < x < 1$. This is the Taylor series about 0 for $f(x) = (1+x)^{-1/2}$.

- (b) Integrating both sides of the equation in part (a) gives

$$\begin{aligned}\int (1+x)^{-1/2} dx \\ = \int \left(1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots\right) dx;\end{aligned}$$

that is,

$$2(1+x)^{1/2} = c + x - \frac{1}{4}x^2 + \frac{1}{8}x^3 - \frac{5}{64}x^4 + \dots,$$

where c is a constant.

Taking $x = 0$ gives $2 \times 1^{1/2} = c$, so $c = 2$.

Therefore

$$\begin{aligned}2(1+x)^{1/2} &= 2 + x - \frac{1}{4}x^2 + \frac{1}{8}x^3 - \frac{5}{64}x^4 + \dots \\ &= 2 \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots\right),\end{aligned}$$

for $-1 < x < 1$.

This is twice the Taylor series about 0 for $f(x) = (1+x)^{1/2}$ found in Exercise 21(a).

Solution to Exercise 32

- (a) Differentiating the Taylor series about 0 for $1/(1+x)^2$ gives

$$-\frac{2}{(1+x)^3} = -2 + 6x - 12x^2 + 20x^3 - \dots,$$

for $-1 < x < 1$. Dividing both sides by -2 gives the required Taylor series

$$\frac{1}{(1+x)^3} = 1 - 3x + 6x^2 - 10x^3 + \dots,$$

for $-1 < x < 1$.

- (b) Integrating the Taylor series about 0 for $1/(1+x)^2$ gives

$$-\frac{1}{(1+x)} = c + x - x^2 + x^3 - x^4 + \dots,$$

for $-1 < x < 1$, where c is a constant.

Taking $x = 0$ gives $-1/1 = c$, so $c = -1$.

Therefore

$$-\frac{1}{(1+x)} = -1 + x - x^2 + x^3 - x^4 + \dots,$$

for $-1 < x < 1$. Multiplying both sides by -1 gives the required Taylor series

$$\frac{1}{(1+x)} = 1 - x + x^2 - x^3 + x^4 - \dots,$$

for $-1 < x < 1$.

(This agrees with the answer to Exercise 30(a).)